Maximal inequalities, optional stopping time theorem

**Def.** Let $(X_t)_{t\in I}$ be a process. The maximal function of $(X_t)$ is $X^*(\omega) := \sup_{t\in I} |X_t(\omega)|$.

**Theorem.** Let $(X_n)_{n\in\{1,2,\dots,N\}}$ be a submartingale

Then $\forall \lambda>0$: $\lambda P(X^*\geq\lambda) \leq \int_{X^*\geq\lambda} |X_N|\, dP$

**Proof.** Observe: $|X_n|$ is a submartingale.

$$T := \begin{cases} \min\{n : |X_n|\geq\lambda\}, & \text{if } X^*\geq\lambda \\ N, & \text{otherwise} \end{cases}$$ - stopping time.

so $T\leq N$, and

$$E(|X_N|) \geq E(|X_T|) = \int_{X^*\geq\lambda} X_T\, dP + \int_{X^*<\lambda} |X_N|\, dP \geq$$

$$\lambda P(X^*\geq\lambda) + \underbrace{\int_{X^*<\lambda} |X_N|\, dP.}_{\text{subtract to get Thm}}$$

**Theorem** Let $p>1$. Let $(X_t)_{t\in I}$ be a submartingale

Assume that $X_t$ has continuous trajectories if $I = \mathbb{R}/\mathbb{R}_+$

Then $E(X^{*p}) \leq \left(\frac{p}{p-1}\right)^p \sup_t E(|X_t|^p)$.

Proof Let us first prove for finite $I = \{1,2,\dots,N\}$.

Let $\mu$ be the law of $X^*$.

Then

$$E(X^{*p}) = \int_0^\infty \lambda^p\, d\mu(\lambda) = \int_0^\infty p\lambda^{p-1} P(X^*\geq\lambda)\, d\lambda \leq$$

$$\int^\infty p\lambda^{p-1}\left(\frac{1}{\lambda}\int |X_N|\, dP\right) d\lambda \overset{\text{Fubini}}{=} p\, E\left(|X_N| \int^{X^*} \lambda^{p-2} d\lambda\right)$$

$$\int_0^\infty p\lambda^{p-1}\left(\frac{1}{\lambda}\int_0 |X_N|\,dP\right)d\lambda \overset{\text{Fubbini}}{=} p\,E\left(|X_N|\int_0^{X^*}\lambda^{p-2}d\lambda\right)$$

$$= \frac{p}{p-1}E\left(|X_N|\left(X^{*(p-1)}\right)\right) \overset{\text{Younge}}{\leq} \frac{p}{p-1}E\left(X^{*p}\right)^{\frac{p-1}{p}}E\left(|X_N|^p\right)^{\frac{1}{p}}$$

For discrete time: let $N\to\infty$, then

$X^{(N)*} := \max_{1\leq n\leq N}|X_n| \nearrow X^*$, and for each $N$

$$E\left(\left(X^{(N)*}\right)^p\right) \leq \left(\frac{p}{p-1}\right)^p E\left(|X_N|^p\right) \leq \left(\frac{p}{p-1}\right)^p \sup_n E\left(|X_n|^p\right)$$

So

$$E\left(X^{*p}\right) = \lim_{N\to\infty} E\left(X^{(N)*p}\right) \leq \left(\frac{p}{p-1}\right)^p \sup_n E\left(|X_n|^p\right).$$

For continuous time, observe that

$X^* = \sup_{t\in\mathbb{Q}}|X_t|$ – by continuity.

And we can represent $\mathbb{Q} = \bigcup D_n$ – finite sets, $D_n \subset D_{n+1}$.

As before, $X^{(n)*} := \max_{t\in D_n}|X_t| \nearrow X^*$, and

$$E\left(X^{*p}\right) = \lim_{n\to\infty} E\left(X^{(n)*p}\right) \leq \left(\frac{p}{p-1}\right)^p \sup_{t\in I} E\left(|X_t|^p\right) \blacksquare$$

**Corollary** In the assumptions of previous corollary,
$$P(X^* \geq \lambda) \leq \frac{\sup_{t\in I} E(|X_t|)}{\lambda}$$

Proof. Discrete approximation, as before.

**Corollary** Let $(X_t)_{t\in I}$ be a submartingale (continuous for $I = \mathbb{R}/\mathbb{R}_+$) and $\sup_t E(|X_t|) < \infty$. Then
$\overline{\lim}_{t\to\infty}|X_t| < \infty$ a.s.

Proof. $P\left(\overline{\lim}\,|X_t| > \frac{1}{\varepsilon}\right) \leq P\left(X^* \geq \frac{1}{\varepsilon}\right) \leq \varepsilon \sup_t E(|X_t|) \blacksquare$

**Thm (Martingale convergence Theorem)**

Let $p>1$, $(X_t)_{t\in I}$ - martingale (continuous in continuous time), $\sup_{t\in I} E(|X_t|^p)<\infty$. Then $\exists\, X_\infty$:

1) $E(|X_\infty|^p)<\infty$, $X_\infty$ is $\mathcal{F}_\infty$-measurable.
2) $X_t \to X_\infty$ a.s. and in $L^p$ (for $p<\infty$).
3) $X_t = E(X_\infty \mid \mathcal{F}_t)$.

Proof. Enough to prove for $p<\infty$ (since $L^\infty \subset L^p$, and a.s. limit of functions from $L^\infty$ is in $L^\infty$).

Take a subsequence $X_{t_k}$ weakly (in $L^p$) converging to some $X$. $X$ - $\mathcal{F}_\infty$-measurable (since so are all $X_{t_k}$). Let $A\in\mathcal{F}_s$ for some $s$. Then $1_A \in L^q$, so
$\int_A X\,dP = \lim_{k\to\infty} \int_\Omega X_{t_k} 1_A\, dP = \lim_{k\to\infty}\int_A X_{t_k}\,dP = \int_A X_s\,dP$, since for $t_k>s$, $E(X_{t_k}\mid\mathcal{F}_s) = X_s$.

So $X_s = E(X\mid \mathcal{F}_s)$. Since $\mathcal{F}_\infty = \sigma(\cup \mathcal{F}_s)$, $X$ is uniquely defined. So, by compactness, $X_t \xrightarrow{w} X$.

A.s. convergence:

Let $\Omega_X(\omega) := \overline{\lim_{t\to\infty}} X_t(\omega) - \underline{\lim_{t\to\infty}} X_t(\omega)$. It is a.s. finite ($\overline{\lim}\,|X_t|<\infty$ a.s.).

Also $P(\Omega_X>\varepsilon) \le P(\overline{\lim}\,|X_t|\ge \frac{\varepsilon}{2}) < \frac{2}{\varepsilon}\int |X_\infty|\,dP$.

Let now $\mathcal{D} := \{g\in L^p : g \text{ is } \mathcal{F}_s\text{-measurable for some } s\}$.

$\mathcal{D}$ is dense in $L^p$.

So $\forall \varepsilon > 0 \quad \exists\, g_\varepsilon : \quad E(|X - g_\varepsilon|^p) < \varepsilon^{2p} \Rightarrow E(|X - g_\varepsilon|) < \varepsilon^2$

Observe; $\Omega_{g_\varepsilon} = 0$ (since $\overline{\lim}\, X_t = \underline{\lim}\, X_t = X_s$), so

$$P(\Omega_X(\omega) > \varepsilon) = P(\Omega_{X - g_\varepsilon}(\omega) > \varepsilon) \le \frac{2}{\varepsilon} \cdot \varepsilon^2 = 2\varepsilon.$$

So $\Omega_X = 0$ a.s., and $\exists \lim_{t \to \infty} X_t =: Y$ a.s.

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$L^p$-convergence: $\| X_t - X\|_p \le \| X_t - g_\varepsilon\|_p + \| g_\varepsilon - X\|_p \underset{t > s}{=}$

$$E\left(E(X - g_\varepsilon | \mathcal{F}_t)^p\right)^{1/p} + \| g_\varepsilon - X\|_p \le 2 \| g_\varepsilon - X\|_p \le 2\varepsilon^2.$$

So $\quad \|X_t - X\|_p \underset{t \to \infty}{\to} 0$.

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Why is $X = Y$? Because $\exists\, X_{t_n} \xrightarrow{a.s} X \Rightarrow X = Y$

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**Def** A family $(X_t)_{t \in I}$ is uniformly integrable if

1) $\forall \varepsilon > 0 \ \exists\, \delta > 0$ : $\forall t \ \forall E \in \mathcal{F}_t$, $P(E) < \delta \Rightarrow \int_E |X_t| dP < \varepsilon$. $\quad g_\varepsilon$

2) $\sup E(|X_t|) < \infty$

(1) $\Rightarrow$ 2) for Lebesgue)

**Observe** $\quad X_t$ - uniformly integrable, by Chebyshev inequality

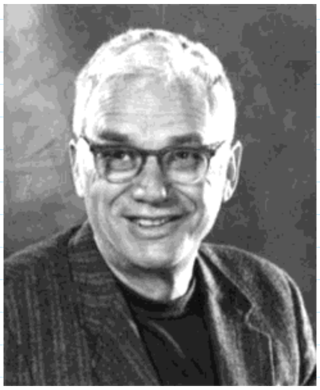
$$\int_E |X_t| dP \overset{\text{Younge}}{\le} \left(\int_E |X_t|^p dP\right)^{1/p} (P(E))^{1/q} \le \delta^{1/q} \sup E(|X_t|^p)^{1/p}.$$

So, by standard convergence theory, $X_t \to Y$ a.s. $\Rightarrow \|X_t - Y\|_1 \to 0$

$$\Big( \int |X_n - Y| dP \le \varepsilon P(\underbrace{|X_n - Y| \le \varepsilon}_{\le \varepsilon}) + \underbrace{\int_{|X_n - Y| > \varepsilon} |X_n - Y| dP}_{\le \varepsilon \text{ for large } n,}$$

since $P(|X_n - Y| \ge \varepsilon) \to 0\Big)$

So, since $\| X_t - X\|_p \to 0$, $X = Y$ ∎



**Joseph L. Doob (1910-2004)**

**Thm** (Doob's martingale convergence).

Let $(X_t)_{t\in I}$ be a submartingale (continuous if $I=\mathbb{R}_+$).

Let $\sup_{t\in I} E(|X_t|)<\infty$. Then $\exists \lim_{t\to\infty} X_t =: X_\infty$ a.s.

(Enough even to assume $\sup E(X_t^+)<\infty$ $(X_t^+=\max(X_t,0))$)

Remark. Same is true for supermartingales $(X_t)$ with $\sup E(X_t^-)<\infty$ (consider submartingale $(-X_t)$).

Proof relies on the following technical lemma.

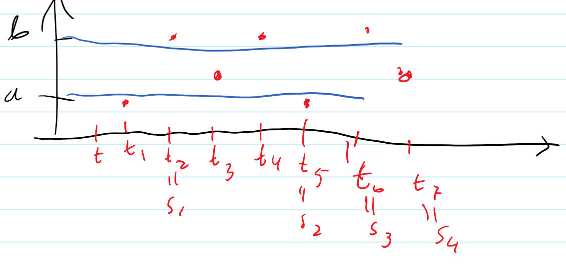
For a function $f:\mathbb{R}_+\to\mathbb{R}_+$ (or $\mathbb{N}\to\mathbb{R}_+$),

let $\{t_1<t_2<\dots<t_d\}=F$ - partition.

$s_1:=\min\{t_i: f(t_i)>b\}$, $s_2:=\min\{t_i>s_1: f(t_i)<a\}$.

$s_{2n+1}:=\min\{t_i>s_{2n}: f(t_i)>b\}$ $\quad s_{2n+2}:=\min\{t_i>s_{2n+1}: f(t_i)<a\}$.

(Convention: $\min\emptyset=t_d$)




$$D(f,[a,b]) := \sup_F \overbrace{\max\{n: s_{2n}<t_d\}}^{D(f,F,[a,b])}$$ - number of crossings from $b$ to $a$.

**Lemma.** Let $(X_t)$ be a submartingale, then

$$\forall a<b,\ (b-a)\ E(D(X,[a,b]) \le \sup_{t\in I} E((X_t-b)^+)$$

$$\left( (X_t-b)^+ := \max(X_t-b,\ 0)\right)$$

<u>Proof.</u> Enough to check $\forall F$.

$s_k$ - stopping times. $A_k := \{\omega : S_k < t_d\} \in \mathcal{F}_{S_k}$.

Moreover, $A_k \supset A_{k+1}$, $X_{S_{2n-1}} > b$ on $A_{2n-1}$, $X_{S_{2n}} < a$ on $A_{2n}$.

Therefore, by finite Optional Stopping Time Theorem:

$$0 \le \int_{A_{2n-1}} (X_{S_{2n-1}} - b)\, dP \overset{\text{submartingale}}{\le} \int_{A_{2n-1}} (X_{S_{2n}} - b)\, dP \le (a-b)\, P(A_{2n}) +$$

$$\int_{A_{2n-1}\setminus A_{2n}} (X_{S_{2n}} - b)\, dP.$$ But on $A_{2n}^c$, we have $S_{2n} = t_d$.

So $(b-a)\, P(A_{2n}) \le \int_{A_{2n-1}\setminus A_{2n}} (X_{t_d} - b)^+\, dP$.

Now observe:

$A_{2n} = \{ D(X, F, [a,b]) > n\}$, $A_{2n} \setminus A_{2n-1}$ - disjoint for different $n$. So just sum up! $\blacksquare$

## Proof of Doob

Assume that $X_t$ is not a.s. convergent as $t\to\infty$

Then $\exists a<b$: $\underline{\lim}\, X_t < a < b < \overline{\lim}\, X_t$ with positive probability. So $D(X,[a,b]) = \infty$ with positive probability: contradiction with the Lemma! $\blacksquare$

We are ready to answer the question: When is $(X_t)$ generated by some $X_\infty$, i.e.

$$X_t = E(X_\infty / \mathcal{F}_t)!$$

**Thm** TFAE equivalent for a martingale $(X_t)_{t\in I}$ (assumed to be continuous when $I = \mathbb{R}/\mathbb{R}_+$).

1) $\exists \lim_{t\to\infty} X_t$ in $L^1$.

2) $\exists X_\infty : E(X_\infty | \mathcal{F}_t) = X_t$.

3) $X_t$ is uniformly integrable.

Proof. 2) $\Longrightarrow$ 3) obvious.

1) $\Longrightarrow$ 2) $E(X_\infty | \mathcal{F}_t) \overset{L^1}{=} \lim_{s\to\infty} E(X_{t+s} | \mathcal{F}_t) = X_t$.

3) $\Longrightarrow$ 1) $\sup_{t\in I} E(|X_t|) < \infty \overset{\text{Doob}}{\Longrightarrow} \exists \lim_{t\to\infty} X_t =: X_\infty$ a.s.

By uniform integrability, $X_t \to X_\infty$ in $L^1$. $\blacksquare$

Thm (General Optional Stopping Time Theorem)

Let $(X_t)$ be a martingale, $S \le T$ - stopping times

1) If $T$ is bounded then $X_S = E(X_T | \mathcal{F}_S)$.

2) If $X$ is uniformly integrable, then

$$X_S = E(X_T | \mathcal{F}_S) = E(X_\infty | \mathcal{F}_S).$$

Reminder. $\mathcal{F}_S := \{A : A \cap \{S \le t\} \in \mathcal{F}_t \ \forall t\}$.

Proof. 2) $\Rightarrow$ 1), since $(X_t)_{t\le M}$ is uniformly integrable, $X_t = E(X_M | \mathcal{F}_t)$.

To prove 2), enough to prove

(*) $X_S = E(X_\infty | \mathcal{F}_S)$, since then

$$X_S = E(E(X_\infty | \mathcal{F}_T) | \mathcal{F}_S) = E(X_T | \mathcal{F}_S).$$

Observe that the family $\left(E(X_\infty \mid \mathcal{B})\right)_{\mathcal{B}-\text{sub-}\sigma\text{-algebra of } \mathcal{F}_\infty}$ is uniformly integrable.

We know (*) for $S_k$ - bounded finitely valued stopping times. Moreover, if $S_k$ has finitely many values $I = \{t_1, t_2, \dots, t_n; \infty\}$, then

$$X_{S_k} = E(X_\infty \mid \mathcal{F}_{S_k}) \quad (\text{this is just a finite-indexed martingale } (X_t)_{t \in I}).$$

Let $S_k$ - finitely valued stopping times, $S_k \downarrow S$.
Then $\mathcal{F}_S \subset \mathcal{F}_{S_k}$. (indeed, $A \cap \{S \le t\} \in \mathcal{F}_t \Rightarrow$

$$A \cap \{S_k \le t\} = \underbrace{A \cap \{S \le t\}}_{\in \mathcal{F}_t} \cap \underbrace{\{S_k \le t\}}_{\in \mathcal{F}_t}).$$

So $\forall A \in \mathcal{F}_S \Rightarrow A \in \mathcal{F}_{S_k}$, so

$$\int_A X_{S_k} \, dP = \int_A X_\infty \, dP \quad (X_{S_k} = E(X_\infty \mid \mathcal{F}_{S_k})).$$

$X_{S_k} \to X_S$, uniformly integrable $\Rightarrow$

$$\forall A \in \mathcal{F}_S \quad \int_A X_S \, dP = \lim_{k \to \infty} \int_A X_{S_k} \, dP = \int_A X_\infty \, dP.$$

So $X_S = E(X_\infty \mid \mathcal{F}_S)$ ■

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**Non-example.** $X_t = \exp(B_t - \frac{t}{2})$ - positive martingale.

$B_t - \frac{t}{2} \to -\infty$ a.s. $\Rightarrow X_t \to 0$ a.s.

Take $T = \inf\{t \ge 0, X_t \le \alpha\}$ $(\alpha < 1)$. Then $T < \infty$ a.s.

$E(X_T) = \alpha \ne E(X_0) = 1$.

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**Def** Stopped martingale.

Let $(X_t)$ - martingale, $T$ - stopping time.

$$X_t^T := \begin{cases} X_t, & t \le T \\ X_T, & t \ge T. \end{cases}$$ with respect to $\mathcal{F}_{t\wedge T} := \mathcal{F}_t \cap \mathcal{F}_T$

Observe: $(X_t^T)$ is a martingale. If $(X_t)$ was uniformly integrable, so is $(X_t^T)$.

Proof. If $(X_t)$ is uniformly integrable then $X_s^T$.

$S := \min(t, T)$ - stopping time, $S \le T$, so, by OST, $\mathcal{F}_S = \mathcal{F}_{t\wedge T}$.

$X_t^T = X_S = E(X_T \mid \mathcal{F}_{t\wedge T})$

so $X_t^T$ is uniformly integrable.

If $(X_t)$ is any martingale, then

for fixed $s$, $(X_t)_{t \le s}$ is uniformly integrable

so $X_t^T = E(X_s^T \mid \mathcal{F}_t)$ if $t < s$, by previous